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AN IMPROVED CHARACTERIZATION OF NORMAL SETS AND SOME COUNTER-EXAMPLES

BY

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ABSTRACT

We strengthen G. Rauzy's characterization of normal sets by observing that the so-called elementary sets are precisely the $F_{\sigma\delta}$ sets. This answers in the affirmative Rauzy's open question: Are finite unions of normal sets necessarily normal? We also generalize the notion and characterization of normal sets from subsets of $\mathbb{R}$ to subsets of $\mathbb{R}^d$. This allows us to answer a question of E. Lesigne and M. Wierdl with the following construction: There exist two sequences of real numbers $U = (u_n)_{n\in\mathbb{N}}$, $V = (v_n)_{n\in\mathbb{N}}$ such that $\alpha U + \beta V = (\alpha u_n + \beta v_n)_{n\in\mathbb{N}}$ is uniformly distributed mod 1 if and only if exactly one of the real numbers α, β vanishes. Additionally, we provide the 'ultimate' counter-example (stronger than that of H. G. Meijer and R. Sattler) to a conjecture of S. Uchiyama by constructing a sequence of integers U which is u.d. in $\mathbb{Z}$ (i.e. u.d. mod k for all $k \in \mathbb{N}$), but such that for all real α, αU is not u.d. mod 1.

1. Introduction and results

A sequence $U = (u_n)_{n\in\mathbb{N}}$ of reals is said to be **uniformly distributed mod 1** if, for $0 \le a < b \le 1$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_{[a,b)}(\{u_n\}) = b - a,$$

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where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. We define the sequence U to be **balanced mod 1** if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i u_n} = 0.$$

This latter notion has the physical interpretation that if a unit of mass is divided equally among N points on the unit circle at the angles $2\pi u_n$ (as n ranges from 1 to N), then in the limit as N tends to infinity, the body will balance on the origin. This definition is motivated by Weyl's criterion for uniform distribution mod 1, which we state as: A sequence U is u.d. mod 1 iff kU is balanced mod 1 for all $k \in \mathbb{N}$. For any sequence U taking values in $\mathbb{R}^d$, define

$$\mathrm{B}(U) = \{\alpha \in \mathbb{R}^d \colon \alpha \cdot U \text{ is u.d. mod } 1\},$$
$$\mathrm{Bal}(U) = \{\alpha \in \mathbb{R}^d \colon \alpha \cdot U \text{ is balanced mod } 1\}.$$

Then Weyl's criterion takes the form

$$\mathrm{B}(U) = \bigcap_{k\in\mathbb{N}} \frac{1}{k} \mathrm{Bal}(U).$$

A set B is called **normal** if there exists a sequence U such that $B = \mathrm{B}(U)$.

THEOREM 1 (Main Theorem): *Given $B \subset \mathbb{R}^d$, there exists a sequence U in $\mathbb{R}^d$ ($\mathbb{Z}^d$) such that $B = \mathrm{Bal}(U)$ iff the following conditions hold:*
(1a) $0 \notin B$.
(1b) $B = -B$ ($B = \mathbb{Z}^d - B$).
(2) B *is* $\mathrm{F}_{\sigma\delta}$.

The necessity of these conditions is easy to verify. We defer the proof of sufficiency until the next section. The following corollaries are immediate:

COROLLARY 1: *B is normal for a sequence in $\mathbb{R}^d$ ($\mathbb{Z}^d$) iff the conditions of Theorem 1 hold, as well as*
(1c) $\mathbb{N}B = B$.

COROLLARY 2: *The class of normal sets is closed under finite unions and countable intersections.*

Now we present several interesting examples which arise from the characterization. The first gives a strong negative answer to the following question of E. Lesigne and M. Wierdl: If U and V are two sequences of real numbers such that αU and αV are u.d. mod 1 for Lebesgue almost all α, does it follow that

$\alpha U + \beta V$ is u.d. mod 1 for a.a. pairs (α, β)? It was in working out the answer to this question that the results of the present paper were obtained. The author is extremely grateful to Michael Boshernitzan, who guided his study of uniform distribution, for giving him this problem and providing invaluable feedback on this work.

Example 1: For every $d > 1$, there exist d sequences of real numbers such that every non-trivial linear combination of fewer than d of the sequences is u.d. mod 1, but any linear combination involving all of the sequences is not u.d. mod 1.

This follows easily from Corollary 1 by taking $B = \{\alpha \in \mathbb{R}^d\colon \exists j,k \text{ s.t. } \alpha_j = 0, \alpha_k \neq 0\}$. Although the set B has quite a simple structure, the example is still somewhat surprising.

Example 2: For every $d > 1$, there exist d sequences of integers such that a linear combination of the sequences is u.d. mod 1 iff one coefficient is integral and another is irrational.

We may see this by applying Theorem 1 to $B = \{\alpha \in \mathbb{R}^d\colon \exists j,k \text{ s.t. } \alpha_j \in \mathbb{Z}, \alpha_k \notin \mathbb{Z}\}$, which gives a sequence U in $\mathbb{Z}^d$ such that $B = \mathrm{Bal}(U)$. It follows that $\mathrm{B}(U) = \{\alpha \in \mathbb{R}^d\colon \exists j,k \text{ s.t. } \alpha_j \in \mathbb{Z}, \alpha_k \notin \mathbb{Q}\}$.

Example 3: There exists a sequence U in $\mathbb{R}^d$ such that $\alpha \cdot U$ is u.d. mod 1 for $\alpha \neq 0$ iff $1, \alpha_1, \ldots, \alpha_d$ are linearly dependent over $\mathbb{Q}$.

This example follows from Corollary 1 by taking B to be the countable union of all rational affine hyper-planes in $\mathbb{R}^d$, less the origin. Rauzy proved this directly (i.e. without using his characterization) in [7], for $d = 1$.

Example 4: The set of Liouville numbers is normal for a sequence of integers. More generally, there exists a sequence U in $\mathbb{Z}^d$ such that $\alpha \cdot U$ is u.d. mod 1 iff α is a Liouville vector—by which we mean for all $C > 0$ and $k \in \mathbb{N}$, there exists $q \in \mathbb{N}$ and $p \in \mathbb{Z}^d$ such that $0 < \|q\alpha - p\|_\infty < Cq^{-k}$.

This follows from Corollary 1 and the fact that the set of Liouville vectors is G_δ.

In the final example, we deal with the following (false) conjecture of S. Uchiyama, from [8]: If U is a sequence of integers which is u.d. in $\mathbb{Z}$, then αU is u.d. mod 1 for a.a. real numbers α. H. G. Meijer, in [3], produced the first counter-example and, with R. Sattler in [4], proved the following stronger one: There exists a sequence U of integers which is u.d. in $\mathbb{Z}$, but such that for a.a. real α, αU is not u.d. mod 1. Our counter-example replaces "almost all" with "all" and generalizes to dimensions greater than 1. We gave a definition for uniform distribution in $\mathbb{Z}$ in the abstract. More generally, a sequence U is said to be u.d. in $\mathbb{Z}^d$ provided its image is u.d. in every finite quotient of $\mathbb{Z}^d$. This is a specific case of the general definition for uniform distribution in a locally compact group, which may be found in [2]. H. Niederreiter studied uniform distribution in $\mathbb{Z}^d$ in [5] and stated a Weyl criterion equivalent to the following: A sequence U is u.d. in $\mathbb{Z}^d$ iff $\mathrm{Bal}(U) \supset \mathbb{Q}^d \setminus \mathbb{Z}^d$.

Example 5: There exists a sequence U of d-tuples of integers which is u.d. in $\mathbb{Z}^d$, but such that $\alpha \cdot U$ fails to be u.d. mod 1 for every $\alpha \in \mathbb{R}^d$.

We obtain U from Theorem 1 by taking $B = \mathbb{Q}^d \setminus \mathbb{Z}^d$. It then follows directly from the Weyl criterion that U is u.d. in $\mathbb{Z}^d$. By our choice of B, we have that $\alpha \cdot U$ fails to be balanced and hence u.d. mod 1 for every $\alpha \in \mathbb{R}^d \setminus (\mathbb{Q}^d \setminus \mathbb{Z}^d)$. But since U takes values in $\mathbb{Z}^d$, it follows that $\alpha \cdot U$ also fails to be u.d. mod 1 for every $\alpha \in \mathbb{Q}^d$, and thus for all α.

2. Proof of the Main Theorem

We begin with Rauzy's original characterization, from [6], which he stated as follows:

THEOREM 2.1 (Rauzy): *$B \subset \mathbb{R}$ is normal iff*

(1) *$0 \notin B$ and $\forall q \in \mathbb{Z}^*, qB \subset B$.*

(2) *There exists a sequence of continuous real-valued functions $(f_N)_{N \in \mathbb{N}}$ such that $x \in B$ iff $f_N(x) \to 0$ as $N \to \infty$.*

Additionally, B is normal for a sequence in $\mathbb{Z}$ iff (1) and (2) hold, as well as

(3) *$\forall g \in \mathbb{Z}, g + B \subset B$.*

Rauzy gave the name **elementary** to sets satisfying condition (2). There are three essential differences between Rauzy's characterization and ours. The first is that we have extended the characterization to subsets of $\mathbb{R}^d$. Secondly, Theorem 1 is stated in terms of the notion "balanced mod 1" instead of "u.d. mod 1," allowing us greater flexibility to deal with other questions of uniform distribution (e.g. Example 5). Finally, "B is elementary" has been replaced with "B is $\mathrm{F}_{\sigma\delta}$." Since only slight changes need be made to Rauzy's original proof in order to obtain Theorem 1, we will describe the ideas of Rauzy's argument and then give the specific modifications.

As remarked earlier, the necessity of the characterizing conditions is easy to check. Rauzy proves sufficiency by, given B satisfying the conditions, constructing a sequence U for which, to put it in our own terms, $B = \mathrm{Bal}(U)$. This is achieved by first choosing a special sequence of continuous functions $(f_N)_{N \in \mathbb{N}}$

such that $f_N(\alpha) \to 0$ precisely at the points $\alpha \in B$, and then picking $U = (u_n)_{n\in\mathbb{N}}$ so that the Weyl averages $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i\alpha u_n}$ have the same point-wise limit behavior. In order to make this work, the functions f_N are chosen to be positive definite ($\hat{f}_N \geq 0$) and normalized ($f_N(0) = \|\hat{f}_N\|_1 = 1$). Under these conditions, each f_N may be uniformly approximated arbitrarily well on any large cube by a trigonometric polynomial having positive coefficients which sum to 1. The sequence U is then chosen in repeated blocks so that the Weyl averages successively approximate each f_N, and making convex transitions in between. The functions f_N are actually taken from the two-parameter family $F_{\varepsilon,\alpha}$ ($\tilde{F}_{\varepsilon,\alpha}$ when integral U is desired) defined below. The key feature of the function $F_{\varepsilon,\alpha}$ is that each has 'bumps' of radius 2ε at the points $0, \alpha, -\alpha$ and equals 0 elsewhere (the functions $\tilde{F}_{\varepsilon,\alpha}$ have this property on $\mathbb{R}^d/\mathbb{Z}^d$. This allows one to 'spoil' $\mathbb{R}^d \setminus B$ using a simple topological argument (requiring that B be elementary) to choose the f_N.

It is quite easy to generalize Rauzy's original functions defined on $\mathbb{R}$ to functions defined on $\mathbb{R}^d$.

For $x \in \mathbb{R}$ and $\varepsilon > 0$, let

$$\varphi_\varepsilon(x) = \begin{cases} 1 - |x|/2\varepsilon & \text{if } |x| \leq 2\varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

and for $x \in \mathbb{R}^d$, let

$$\varphi_\varepsilon(x) = \prod_{j=1}^{d} \varphi_\varepsilon(x_j).$$

It is easy to check that

$$\varphi_\varepsilon(x) = \int_{\mathbb{R}^d} \left(\prod_{j=1}^{d} 2\varepsilon \left(\frac{\sin 2\pi\varepsilon\xi_j}{2\pi\varepsilon\xi_j} \right)^2 \right) e^{2\pi i x\cdot\xi} d\xi.$$

Then setting

$$G_{\varepsilon,\alpha}(x) = \varphi_\varepsilon(x) + \tfrac{1}{2}\left(\varphi_\varepsilon(x-\alpha) + \varphi_\varepsilon(x+\alpha)\right),$$

we have

$$G_{\varepsilon,\alpha}(x) = \int_{\mathbb{R}^d} \left(\prod_{j=1}^{d} 2\varepsilon \left(\frac{\sin 2\pi\varepsilon\xi_j}{2\pi\varepsilon\xi_j} \right)^2 \right) (1 + \cos 2\pi\alpha\cdot\xi) e^{2\pi i x\cdot\xi} d\xi.$$

Finally, let

$$F_{\varepsilon,\alpha} = G_{\varepsilon,\alpha}/G_{\varepsilon,\alpha}(0).$$

Each $F_{\varepsilon,\alpha}$ is a normalized positive definite function with the properties

$$\min\{\|x-\alpha\|_\infty, \|x\|_\infty, \|x+\alpha\|_\infty\} \le \varepsilon \implies F_{\varepsilon,\alpha}(x) \ge 2^{-d-2},$$
$$\min\{\|x-\alpha\|_\infty, \|x\|_\infty, \|x+\alpha\|_\infty\} \ge 2\varepsilon \implies F_{\varepsilon,\alpha}(x) = 0.$$

The functions $F_{\varepsilon,\alpha}$ have periodic analogs $\tilde{F}_{\varepsilon,\alpha}$ defined by

$$\tilde{G}_{\varepsilon,\alpha}(x) = \sum_{g\in\mathbb{Z}^d} G_{\varepsilon,\alpha}(g+x),$$
$$\tilde{F}_{\varepsilon,\alpha} = \tilde{G}_{\varepsilon,\alpha}/\tilde{G}_{\varepsilon,\alpha}(0).$$

These definitions are all that is needed to extend Rauzy's proof to higher dimensions. Also, we may state the characterization in terms of the notion "balanced mod 1," since this is how Rauzy structured his argument. Therefore the only thing left to show is that "elementary" may be replaced by "$\mathrm{F}_{\sigma\delta}$." Rauzy himself noted that all elementary sets are $\mathrm{F}_{\sigma\delta}$. He also proved that the class of elementary sets is closed under countable intersections. Furthermore, he showed that open sets and 'mostly disjoint' countable unions of closed sets are elementary. It turns out, though, that H. Hahn had proved the following theorem long before (see 23.18 and 23.20 in [1]). The author is indebted to Y. Peres for correctly guessing that he had needlessly re-proved this classical result and for tracking down a reference (with the help of R. Lyons and S. Solecki).

THEOREM 2.2 (Hahn): *For $E \subset \mathbb{R}^d$, the following are equivalent:*

(1) $\exists$(*continuous* $f_N\colon \mathbb{R}^d \to \mathbb{R})_{N\in\mathbb{N}}$ *s.t.* $E = \{x \in \mathbb{R}^d\colon \lim_{N\to\infty} f_N(x)$ *exists*$\}$.

(2) $\exists$(*continuous* $f_N\colon \mathbb{R}^d \to \mathbb{R})_{N\in\mathbb{N}}$ *s.t.* $E = \{x \in \mathbb{R}^d\colon \lim_{N\to\infty} f_N(x) = 0\}$.

(3) E *is* $\mathrm{F}_{\sigma\delta}$.

3. Zero-one law for normal sets

It is easy to show that if U is a sequence in $\mathbb{Z}^d$, then $\mathrm{B}(U) \cap [0,1)^d$ has Lebesgue measure either zero or one. It follows, in this case, that either $\mathrm{B}(U)$ or its complement in $\mathbb{R}^d$ has Lebesgue measure zero. This can fail for non-integral sequences, as may be seen by applying Corollary 1 to $B = \{x \in \mathbb{R}\colon |x| > 1\}$. It is always true, however, for a sequence U in $\mathbb{R}$, that $B = \mathrm{B}(U)$ has either Lebesgue measure zero or Banach density one. A proof follows. The only properties of B we use are that $B \subset \mathbb{R}$ is measurable and $\mathbb{Z}^* B = B$.

We denote Lebesgue measure by λ. Suppose $\lambda(B) > 0$. We will show that

$$\liminf_{T-S\to\infty} \frac{\lambda(B\cap(S,T))}{T-S} = 1.$$

Fix $0 < t < 1$, and let $x > 0$ be a Lebesgue density point of B (then so is $-x$). Choose $N \in \mathbb{N}$ such that for $0 < \varepsilon \le x/N$, $\lambda(B \cap (x-\varepsilon, x+\varepsilon)) \ge 2\varepsilon\sqrt{t}$. Then for all $M \ge N$ and $0 < r \le x$, $\lambda(B \cap (Mx - r, Mx + r)) \ge 2r\sqrt{t}$. Therefore

$$T - S \ge \frac{2Nx}{1-\sqrt{t}} \Longrightarrow \frac{\lambda(B \cap (S,T))}{T-S} \ge t.$$

For a sequence U taking values in $\mathbb{R}^d$, the only generalization we can make is that $\mathrm{B}(U)$ restricted to any line L through the origin has either Lebesgue measure zero or Banach density one in L. Additionally, if U takes values in $\mathbb{Z}^d$, then for any rational subspace S of $\mathbb{R}^d$, either $\mathrm{B}(U)$ restricted to S or its complement in S has Lebesgue measure zero.

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